Extremes for metocean: Recent developments and take-aways

Philip Jonathan

Lancaster University

IOGP, London, September 2026
(slides at *www.lancs.ac.uk/∼jonathan)*

Overview of masterclass

- A: Fundamentals
- B: Covariates
- C: Multivariate
- D: covXtreme
- E: Recent developments and take-aways

Overview of E: Recent developments and take-aways

Recent developments

- Full probabilistic design
- Model of within-storm dependence of sea states
- Angular-radial model for multivariate extremes
- Return values with uncertainty

Take-aways

- Key things to remember
- Best practice mantra

Full probabilistic design

Set-up

- Storm storm peak events $\boldsymbol{X}^{\text{sp}}$ dependent on covariates $\boldsymbol{\Theta}^{\text{sp}}$
- An evolving within-storm environment $\{(\mathbf{X}_s, \boldsymbol{\Theta}_s)\}_{s \in S_T}$ for storm of random length T
- Structural "loading" $\boldsymbol{Y}$
- Everything subject to sources of uncertainty $\boldsymbol{Z}$
- $\boldsymbol{Z}$, $\boldsymbol{\Theta}^{\text{sp}}$, $\boldsymbol{X}^{\text{sp}}$, $\{(\mathbf{X}_s, \boldsymbol{\Theta}_s)\}_{s \in S_T}$ and $\boldsymbol{Y}$ are multidimensional

Full probabilistic design

Unconditional distribution of loading for a random storm

$$
\begin{aligned}
F_{Y}(y) \quad = \quad & \int_{\zeta} \int_{(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau)} \int_{\mathbf{x}^{\text{sp}}} \int_{\boldsymbol{\theta}^{\text{sp}}} \\
\times \quad & F_{Y|\{(\mathbf{X}_s,\boldsymbol{\Theta}_s)\}_{s\in S_T},\mathbf{Z}}(y|\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in S_\tau},\boldsymbol{\zeta}) \\
\times \quad & f_{(\{(\mathbf{X}_s,\boldsymbol{\Theta}_s)\}_{s\in\mathcal{S}_T},T)|\mathbf{X}^{\text{sp}},\boldsymbol{\Theta}^{\text{sp}},\mathbf{Z}}\left(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau \mid \mathbf{x}^{\text{sp}},\boldsymbol{\theta}^{\text{sp}},\boldsymbol{\zeta}\right) \\
\times \quad & f_{\mathbf{X}^{\text{sp}}|\boldsymbol{\Theta}^{\text{sp}},\mathbf{Z}}(\mathbf{x}^{\text{sp}}|\boldsymbol{\theta}^{\text{sp}},\boldsymbol{\zeta}) \\
\times \quad & f_{\boldsymbol{\Theta}^{\text{sp}}|\mathbf{Z}}(\boldsymbol{\theta}^{\text{sp}}|\boldsymbol{\zeta}) \\
\times \quad & f_{\mathbf{Z}}(\boldsymbol{\zeta}) \\
\times \quad & d\boldsymbol{\theta}^{\text{sp}}\, d\mathbf{x}^{\text{sp}}\, d(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau)\, d\boldsymbol{\zeta}
\end{aligned}
$$

Full probabilistic design

Distribution of annual maximum loading

- For univariate load Y here

$$F_A(y) \quad = \quad \int_c [F_Y(y)]^c \, f_C(c) \, dc$$

- Density f_C of annual rate of occurrence of storms
- Return value for return period P years given by $F_A^{-1}(1 - 1/P)$

Expected annual utility for independent storms

- For multivariate load $\boldsymbol{Y}$ here

$$\mathbb{E}(U|\boldsymbol{\mathcal{R}}) \quad = \quad \int_{\boldsymbol{y}} U(\boldsymbol{y}|\boldsymbol{\mathcal{R}}) f_A(\boldsymbol{y}) \, d\boldsymbol{y}$$

- System utility $U(\boldsymbol{y}|\boldsymbol{\mathcal{R}})$ given system "strength" characteristics $\boldsymbol{\mathcal{R}}$
- Solve for $\boldsymbol{\mathcal{R}}$ to achieve required expected annual utility

Full probabilistic design

Expected annual utility for dependent storms

- For year with M random storms, M random

$$
\begin{aligned}
\mathbb{E}(U_A|\boldsymbol{\mathcal{R}}) &= \int_m \int_{\boldsymbol{y}_1} \dots \int_{\boldsymbol{y}_m} U_A(\boldsymbol{y}_1, \dots, \boldsymbol{y}_m|\boldsymbol{\mathcal{R}}) \\
&\times f_{\boldsymbol{Y}_1,\dots,\boldsymbol{Y}_m,M}(\boldsymbol{y}_1, \dots, \boldsymbol{y}_m, m) \\
&\times d(\boldsymbol{y}_1, \dots, \boldsymbol{y}_m, m)
\end{aligned}
$$

- System annual utility $U_A(\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_m|\boldsymbol{\mathcal{R}})$ given system "strength" characteristics $\boldsymbol{\mathcal{R}}$
- $f_{\boldsymbol{Y}_1,\dots,\boldsymbol{Y}_m,M}$ is the joint density of multivariate loading from M random storms
- Solve for $\boldsymbol{\mathcal{R}}$ to achieve required expected annual utility

Full probabilistic design

The quantities we need to estimate $F_{\boldsymbol{Y}}$

- $f_{\boldsymbol{Z}}$: a means of quantifying uncertainty; random thresholds, bootstrap resampling
- $f_{\boldsymbol{\Theta}^{\text{sp}}|\boldsymbol{Z}}$: a model for rate of occurrence of storms with covariates
- $f_{\boldsymbol{X}^{\text{sp}}|\boldsymbol{\Theta}^{\text{sp}},\boldsymbol{Z}}$: a model for storm peaks
- $f_{(\{(\boldsymbol{X}_s,\boldsymbol{\Theta}_s)\}_{s\in\mathcal{S}_T},T)|\boldsymbol{X}^{\text{sp}},\boldsymbol{\Theta}^{\text{sp}},\boldsymbol{Z}}$: a model for within storm evolution

- We can estimate all these using the statistical models discussed here
- See Speers et al. [2024], Speers et al. [2026]

Model of within-storm dependence of sea states

Canonical extensions of the conditional extremes model

- Basic: $X|(Y = y), y > u$
- Temporal: "storm evolution" $X_1, X_2, ..., X_\tau|(X_0 = x_0), x_0 > u$
- Spatial: "spatial dependence" $X_1, X_2, ..., X_s|(X_0 = x_0), x_0 > u$

Idea

$$X_1, X_2, ..., X_p|(Y = y) = \boldsymbol{\alpha} y + y^{\boldsymbol{\beta}} \boldsymbol{Z}$$

- Impose appropriate structure on parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and distribution of $\boldsymbol{Z}$
  - e.g. $\boldsymbol{\alpha}$ evolves smoothly in time
  - e.g. $\boldsymbol{Z}$ follows a multivariate Gaussian or extension thereof with appropriate mean and covariance forms
- Make a simplifying assumption
  - e.g. $X_{t+1}, X_{t+2}|(X_t = x) = [\alpha_1 \alpha_2]x + x^{[\beta_1 \beta_2]}[Z_1 Z_2]$

Further extensions

- Non-stationary and multivariate temporal and spatial models

Model of within-storm dependence of sea states

Applied conditional extremes references

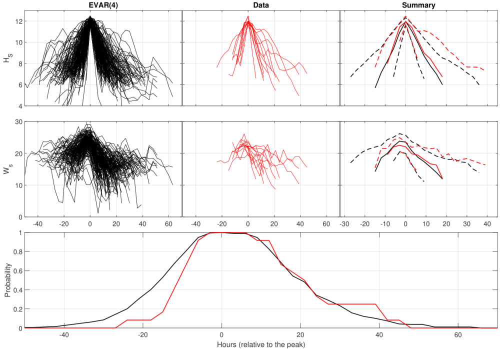
- Non-stationary : Jonathan et al. [2014]
- Time-series : Winter and Tawn [2016], Tendijck et al. [2019], Tendijck et al. [2024a]
- Mixture model : Tendijck et al. [2023]
- Spatial : Shooter et al. [2021b], Shooter et al. [2021a], Shooter et al. [2022]

Model of within-storm dependence of sea states

Extremal vector auto-regression (EVAR; Tendijck et al. 2024a)

- On Laplace margins, with component-wise operations and $\boldsymbol{X}_t \in \mathbb{R}^d$:

$$\boldsymbol{X}_{t+k} \,|\, (\boldsymbol{X}_t, \ldots, \boldsymbol{X}_{t+k-1}, X_{t,1} = y) = \sum_{\ell=1}^{k} \boldsymbol{A}_\ell \boldsymbol{X}_{t+k-\ell} + y^{\boldsymbol{b}} \boldsymbol{Z}, \quad y > u \gg 0$$



Excursions of H_S (top) and W_S (middle) from EVAR(4) model (left; black), observed (middle; red) on original margins with storm peak $H_S \in [11.5, 12.5]$; right-hand plots summarise the observed (red) and EVAR(4) (black) excursions, using median (solid), 10% and 90% quantiles (dashed). In the bottom panel, we plot survival probabilities for observed (red) and EVAR(4) (black) excursions relative to the time of the excursion maximum.

Angular-radial model for multivariate extremes

Semi-parametric angular-radial (SPAR) model

- Radial R and angular Q components. Then joint density factorised as

$$f_{R,Q}(r,q) = f_Q(q) f_{R|Q}(r|q)$$

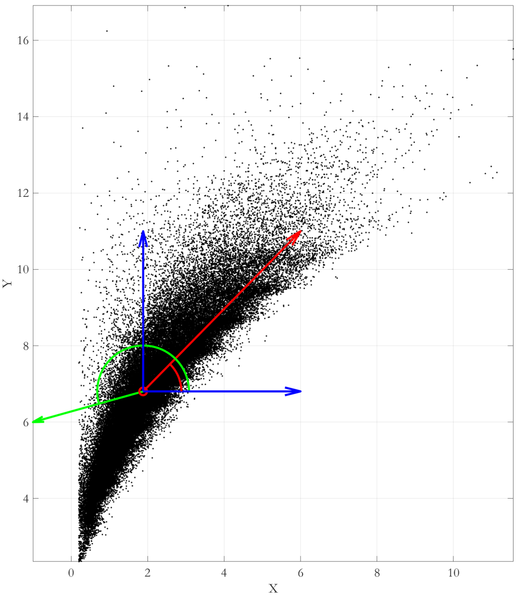
- Assume GP conditional tail for $R|(Q=q)$, with parameters varying smoothly with angle q above some threshold $\psi(q)$ with non-exceedance probability $\tau(q)$

$$f_{R,Q}(r,q) = f_Q(q) \times \tau(q) f_{GP}(r - \psi(q)|\xi(q), \sigma(q)), \quad r > \psi(q)$$

  with smoothly varying $\psi(q)$, $\tau(q)$, $\xi(q)$ and $\sigma(q)$. Also assume angular density $f_Q(q)$ varies smoothly with q

- SPAR representation shown to provide good approximations to a large set of copula functions on standard margins
- Is transformation to standard margins necessary?
- Different possible angular-radial decompositions using "generalised co-ordinates"
- $\Rightarrow$ multivariate extremes is just "non-stationary univariate" extremes

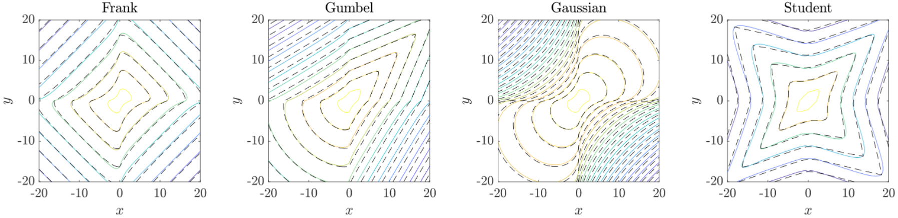
Angular-radial model for multivariate extremes



- SPAR 2-D
- H_S (X) and T_P (Y)
- Radial and angular (R, Q) instead of Cartesian (X, Y)
- OK for all $Q \in [0, 360)$
- OK in higher dimensions
- Algorithms in 2-D same as for directional extremes!

Angular-radial model for multivariate extremes

SPAR fits to extreme value copulas

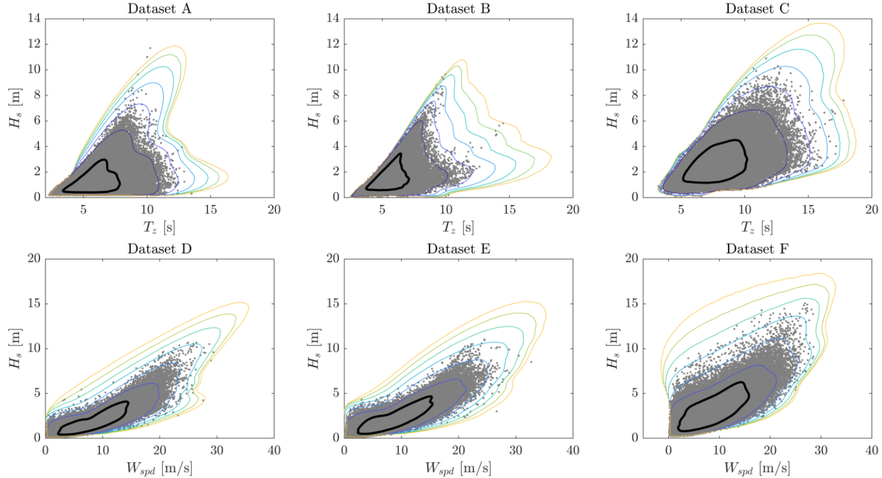


Density contours of various copulas on Laplace margins. All copulas have Pearson correlation coefficient 0.6. Student-t copula has two degrees of freedom. Solid lines: true contours at logarithmic increments. Dashed lines: SPAR-estimated contours.

- SPAR admits AI (e.g. upper tails of Frank and Gaussian) and AD (e.g. upper tails of Gumbel and Student-t)
- SPAR handles all directions (not just “first quadrant”)
- Link to limit sets (see Multivariate slides)

Angular-radial model for multivariate extremes

Density contours from SPAR fits to data



Density contours from SPAR model for 6 samples.

Angular-radial model for multivariate extremes

SPAR references

- Mackay and Jonathan [2023] (original work on arXiv)
- Murphy-Barltrop et al. [2025] (first peer-reviewed publication)
- Mackay et al. [2025] (5-D SPAR with deep learning)

- Parallel developments in the statistics literature

Return values with uncertainty

What is a return value?

- Random variable A represents the maximum value of some physical quantity X per annum
- Forget about all complicating issues like serial dependence, covariates and other sources of dependence and uncertainty
- The P-year return value x_P of X is then defined by the equation

$$F_A(x_P) = \Pr(A \leq x_P) = 1 - \frac{1}{P}$$

- Or

$$x_P = F_A^{-1}(1 - \frac{1}{P})$$

- Typically $P \in [10^2, 10^8]$ years

Return values with uncertainty

An alternative definition

- Random variable A_P represents the P-year maximum value of X
- The P-year return value x'_P of X can be found from F_{A_P} for large P, assuming independent annual maxima since

$$\begin{aligned} F_A(x_P) &= 1 - \frac{1}{P} \\ \Rightarrow F_{A_P}(x_P) &= \left(1 - \frac{1}{P}\right)^P \approx \exp(-1) \end{aligned}$$

- Use $F_{A_P}(x'_P) = \exp(-1)$ to define an alternative return value x'_P

Return values with uncertainty

Estimating a return value

- To estimate x_P, we need knowledge of the distribution function F_A of the annual maximum
- We might estimate F_A using extreme value analysis on a sample of independent observations of A
- Typically more efficient to estimate the distribution $F_{X|X>u}$ of threshold exceedances of X above some high threshold u using a sample of independent observations of X, and use this in turn to estimate F_A and x_P
- For threshold u, estimate GP parameter ξ, σ, Poisson annual rate of occurrence of exceedances λ
- So what's the problem?

Return values with uncertainty

Parameter uncertainty

- x_p can be estimated easily in the absence of uncertainty
- In reality, we estimate parameters λ, ψ, σ and ξ from a sample of data, and we cannot know their values exactly
- How does this epistemic uncertainty affect return value estimates?
- A number of different plausible estimators for return values under uncertainty
- Different estimators perform differently (bias and variance)
- Which estimators are likely to perform reasonably in fairly general circumstances?
- Is it even sensible or desirable to estimate return values?

Return values with uncertainty

Predictive distributions

- If a distribution $F_{Y|\boldsymbol{Z}}$ of random variable Y is known conditional on random variables $\boldsymbol{Z}$, and the joint density $f_{\boldsymbol{Z}}$ of $\boldsymbol{Z}$ is also known, the unconditional predictive distribution $\tilde{F}_Y$ can be evaluated using

$$\tilde{F}_Y(y) = \int_\zeta F_{Y|\boldsymbol{Z}}(x|\zeta) f_{\boldsymbol{Z}}(\zeta)\, d\zeta$$

- Th expected value of deterministic function g of parameters $\boldsymbol{Z}$ given joint density $f_{\boldsymbol{Z}}$ is

$$E[g(\boldsymbol{Z})] = \int_\zeta g(\zeta) f_{\boldsymbol{Z}}(\zeta)\, d\zeta$$

- $\zeta = (\lambda, \psi, \sigma, \xi)$, $Y = A$ (or $Y = A_P$)
- In a Bayesian setting, $f_{\boldsymbol{Z}}(\zeta)$ can be interpreted as the posterior distribution ($\triangleq f_{\boldsymbol{Z}|D}(\zeta)$) of model parameters following a Bayesian inference, influenced by choice of (subjective) prior

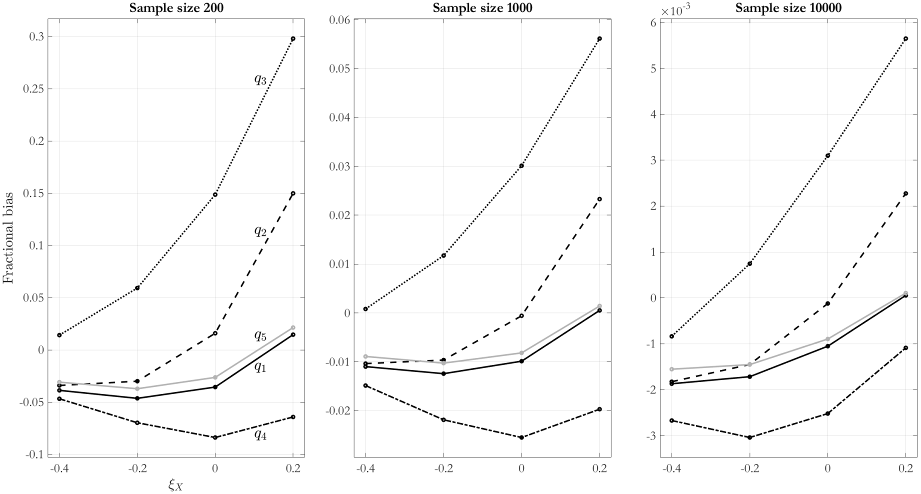
Return values with uncertainty

Different estimators of return value

$$
\begin{aligned}
q_1 &= F^{-1}_{A|\mathbf{Z}}(1 - 1/P \mid \mathbb{E}_{\mathbf{Z}}[\mathbf{Z}]) = F^{-1}_{A|\mathbf{Z}}(1 - 1/P \mid \int_\zeta \zeta f_{\mathbf{Z}}(\zeta) d\zeta) \\
q_2 &= \mathbb{E}_{\mathbf{Z}}[F^{-1}_{A|\mathbf{Z}}(1 - 1/P \mid \mathbf{Z})] = \int_\zeta F^{-1}_{A|\mathbf{Z}}(1 - 1/P \mid \zeta) f_{\mathbf{Z}}(\zeta) d\zeta \\
q_3 &= \tilde{F}^{-1}_{A}(1 - 1/P) \text{ where } \tilde{F}_A(x) = \int_\zeta F_{A|\mathbf{Z}}(x \mid \zeta) f_{\mathbf{Z}}(\zeta) d\zeta \\
q_4 &= \tilde{F}^{-1}_{A_P}(\exp(-1)) \text{ where } \tilde{F}_{A_P}(x) = \tilde{F}^{P}_{A}(x) \\
q_5 &= \text{med}_{\mathbf{Z}}[F^{-1}_{A|\mathbf{Z}}(1 - 1/P \mid \mathbf{Z})]
\end{aligned}
$$

- For inference from large samples, estimators provide similar estimates
- For small samples, estimates can be very different
  - Even when using the same statistical model, and identical data
- Be careful in comparing return value estimates from different sources

Return values with uncertainty



Devil in the detail: fractional bias from 5 estimators for return value.
Posterior predictive (q_3) over-estimates for small samples. Median (q_5) reasonable. See Towe et al. [2023]

Return values with uncertainty

References

- Jonathan et al. [2021] (for stationary marginal extremes)
- Towe et al. [2023] (for associated values from conditional extremes)
- Tendijck et al. [2024b] (for Shell CEVA software)

Wrap-up: cheat sheet

- Estimating tails of distributions is challenging, lots of approaches
  - If you're unsure what you're doing, then try multiple models
  - If inferences about the target diverge $\Rightarrow$ **RED FLAG**
- Doing inference on output of relevant (physical or statistical) simulation models is a rigorous way to evaluate different model choices
- Incorporate physical knowledge
  - Does a modelling assumption make sense from an engineering perspective?
  - Does a modelling assumption make sense statistically?
- Always perform sensitivity analysis with respect to model fiddle factors
  - If large variability in target inferences $\Rightarrow$ **RED FLAG**
  - If possible, use cross-validation to estimate optimal fiddle factor values
- Produce key diagnostic plots for any model you consider
- Incorporate uncertainty from statistical inference in decision making
  - Bootstrapping, model averaging all easy; no excuses
  - Bayesian methods provide consistent framework but steeper learning curve
- Essential to make analysis genuinely reproducible by others in future
  - Document reasoning, choices, assumptions, model performance
- Small samples require simple models
  - Good models default to simple ones when no evidence for complexity
- Full probabilistic design is the future

Wrap-up: cheat sheet

Covariates

- Directional models essential to estimate directional extremes consistently
- Non-stationary extreme value models can get complicated very quickly
  - But good modelling frameworks default to stationary models when no evidence for covariate effects
- Always benchmark against thoughtful alternatives (e.g. covXtreme)

Multivariate

- Applied multivariate extreme value analysis is less well developed
  - Lots of options involving strong assumptions (AD versus AI)
- Conditional extremes more general, simple, and used in other fields
- Emerging angular-radial models like SPAR (admitting AD and AI)

Wrap-up: best practice mantra?

Right balance of good engineering and good statistics
(Joined-up marginal and dependence modelling, aim for probabilistic design, especially for high-risk applications)

Empirical inference demands thorough uncertainty assessment and propagation throughout analysis procedure
(Cross-validation, bootstrapping, model averaging; or Bayesian inference)

Reproducible analysis as rational basis for learning
(Audit trail documenting key assumptions, equations and diagnostic results)

All models are wrong, some models are useful

Diolch & thank-you!

References

P. Jonathan, K. C. Ewans, and D. Randell. Non-stationary conditional extremes of northern North Sea storm characteristics. *Environmetrics*, 25:172–188, 2014.

P. Jonathan, D. Randell, J. Wadsworth, and J.A. Tawn. Uncertainties in return values from extreme value analysis of peaks over threshold using the generalised Pareto distribution. *Ocean Eng.*, 220:107725, 2021.

E Mackay and P Jonathan. Modelling multivariate extremes through angular-radial decomposition of the density function. *(arXiv preprint arXiv:2310.12711)*, 2023.

E. Mackay, C. J. R. Murphy-Barltrop, J. Richards, and P. Jonathan. Deep learning joint extremes of metocean variables using the SPAR model. *J. Offshore Mech. Arct. Eng.*, 148:021201, 2025.

C. J. R. Murphy-Barltrop, E. Mackay, and P. Jonathan. Inference for bivariate extremes via a semi-parametric angular-radial model. *Extremes*, 28:209–238, 2025.

R. Shooter, E Ross, A. Ribal, I. R. Young, and P. Jonathan. Spatial conditional extremes for significant wave height from satellite altimetry. *Environmetrics*, 32:e2674, 2021a.

R Shooter, J A Tawn, E Ross, and P Jonathan. Basin-wide spatial conditional extremes for severe ocean storms. *Extremes*, 24: 241–265, 2021b.

R. Shooter, E Ross, A. Ribal, I. R. Young, and P. Jonathan. Multivariate spatial conditional extremes for extreme ocean environments. *Ocean Eng.*, 247:110647, 2022.

M. Speers, D. Randell, J. A. Tawn, and P. Jonathan. Estimating metocean environments associated with extreme structural response to demonstrate the dangers of environmental contour methods. *Ocean Eng.*, 311:118754, 2024.

M. Speers, J. Tawn, and P. Jonathan. Sequential design for the efficient estimation of offshore structure failure probability. *Ocean Eng.*, 349:123990, 2026.

S. Tendijck, E. Ross, D. Randell, and P. Jonathan. A non-stationary statistical model for the evolution of extreme storm events. *Environmetrics*, 30:e2541, 2019.

S. Tendijck, E. Eastoe, J. Tawn, D. Randell, and P. Jonathan. Modeling the extremes of bivariate mixture distributions with application to oceanographic data. *J. Am. Statist. Soc.*, 118:1373–1384, 2023.

S. Tendijck, P. Jonathan, D. Randell, and J. A. Tawn. Temporal evolution of the extreme excursions of multivariate kth order Markov processes with application to oceanographic data. *Environmetrics*, 35:e2834, 2024a.

S. Tendijck, D. Randell, G. Feld, and P. Jonathan. Practical non-stationary extreme value analysis of peaks over threshold using the generalised pareto distribution: Estimating uncertainties in return values. *Ocean Eng.*, 312:119247, 2024b.

R. Towe, D. Randell, J. Kensler, G. Feld, and P. Jonathan. Estimation of associated values from conditional extreme value models. *Ocean Eng.*, 272:113808, 2023.

H. C. Winter and J. A. Tawn. Modelling heatwaves in central France: a case-study in extremal dependence. *J. Roy. Statist. Soc. C*, 65:345–365, 2016.